Boundary value problems for holomorphic functions on Lipschitz planar domains

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Abstract

We study the $\bar{\partial}$ equation subject to various boundary value conditions on bounded simply connected Lipschitz domains $D \subset \mathbb{C}$: for the Dirichlet problem with datum in $L^p(bD, \sigma)$, this is simply a restatement of the fact that members of the holomorphic Hardy spaces are uniquely and completely determined by their boundary values. Here we identify the maximal data spaces and obtain estimates in the maximal p-range for the Dirichlet, Regularity-for-Dirichlet, Neumann, and Robin boundary conditions for $\bar{\partial}$.

1   Introduction

Let D be a bounded, rectifiable, simply connected domain in $\mathbb{C}$ whose boundary is endowed with the induced Lebesgue measure σ. We denote by $\mathcal{E}^p(D)$ the Smirnov class

$$\mathcal{E}^p(D) := \left\{ F \in \vartheta(D) : \|F\|^p_{\mathcal{E}^p(D)} := \sup_{j \in \mathbb{N}} \int_{bD_j} |F(\zeta)|^p d\sigma_j(\zeta) < \infty \right\}, \quad 0 < p < \infty,$$

(with the standard modification for $p = \infty$) where $\vartheta(D)$ is the set of holomorphic functions on D, and $\{D_j\}_{j \in \mathbb{N}}$ is (any) exhaustion of D by rectifiable subdomains. Each $F \in \mathcal{E}^p(D)$ has a non-tangential boundary value $\dot{F}(\zeta)$ at σ-a.e. $\zeta \in bD$, see (22). As is well known, such boundary values determine a proper subspace of the Lebesgue space $L^p(bD, \sigma)$, see [12, Theorem 10.3]. The following congruence of Banach spaces (that is, set identity and norm equivalence) was proved by D. Jerison and C. Kenig [16] whenever D is a chord-arc domain:

$$\mathcal{E}^p(D) = \mathcal{H}^p(D), \quad 0 < p \leq \infty. \tag{1}$$

Here $\mathcal{H}^p(D)$ is the holomorphic Hardy space:

$$\mathcal{H}^p(D) := \{F \in \vartheta(D) : F^* \in L^p(bD, \sigma)\} \quad \text{with norm} \quad \|F\|_{\mathcal{H}^p(D)} := \|\dot{F}\|_{L^p(bD, \sigma)} \qquad (2)$$

where F^* denotes the non-tangential maximal function of F, see (21). In view of the congruence (1), we henceforth adopt the notation $\mathcal{H}^p(D)$ and $h^p(bD)$, respectively, for the interior and boundary values of such spaces; hence

$$h^p(bD) := \left\{\dot{F} \ : \ F \in \mathcal{H}^p(D)\right\} \subsetneq L^p(bD, \sigma). \qquad (3)$$

One way to understand the relationship between $\mathcal{H}^p(D)$ and $h^p(bD)$ is to reframe both spaces within the context of a **Dirichlet boundary value problem for** $\bar{\partial}$ with datum in $L^p(bD, \sigma)$, namely

$$\begin{cases} \bar{\partial}F(z) & = \ 0 & z \in D; \\[2mm] \dot{F}(\zeta) & = \ f(\zeta) & \sigma\text{-a.e. } \zeta \in bD; \quad \text{where we are given} \quad f \in L^p(bD, \sigma) \quad (4) \\[2mm] F^* & \in \ L^p(bD, \sigma). \end{cases}$$

This problem is uniquely solvable if and only if f belongs to $h^p(bD)$: in such case, the solution F must lie in $\mathcal{H}^p(D)$. Put another way, we have that $h^p(bD)$ is the maximal data set for the problem (4), and $\mathcal{H}^p(D)$ is the solution space. See also [5] for a related result in the case when $D = \mathbb{D}$ (the unit disk).

If D is a Lipschitz domain (that is, the topological boundary bD is locally the graph of a Lipschitz function, see Definition 2.1), the above fact is stated more precisely as

Theorem 1.1. *Let $D \subset \mathbb{C}$ be a bounded simply connected Lipschitz domain and let $p > 0$. Given $f \in L^p(bD, \sigma)$, we have that (4) is solvable if and only if $f \in h^p(bD)$, in which case (4) has a unique solution and it lies in $\mathcal{H}^p(D)$. Furthermore,*

- *if $p \geq 1$ and $f \in h^p(bD)$, then the solution F admits the representation*

$$F(z) = \mathbf{C}_D f(z), \quad z \in D$$

   where $\mathbf{C}_D f$ is the Cauchy integral for D acting on f;

- *if $1 < p < \infty$ and $f \in h^p(bD)$, we also have that*

$$\|F^*\|_{L^p(bD, \sigma)} \lesssim \|f\|_{L^p(bD, \sigma)} \qquad (5)$$

   and the implied constant depends only on D and p.

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Readers familiar with the analogous result for the Laplace operator might be surprised to see that conclusion (5) in Theorem 1.1 holds for p in the full range: $1 < p < \infty$ rather than $2 - \delta < p < \infty$, see (23) and Theorem 2.4 below. This reflects the fact that the effective data set for (4), namely $h^p(bD)$, is a very small portion of $L^p(bD, \sigma)$ (the data set for (23)). It follows from Theorem 1.1 that if the boundary data of the Dirichlet problem for Laplace operator (23) is taken in $h^p(bD, \sigma)$, then the existence of a unique solution (with estimates) is guaranteed for any $1 < p < \infty$. Analogous statements are true for all the boundary conditions considered below.

Here we seek to determine the maximal data sets for the **Regularity; Neumann, and Robin problems** for $\bar{\partial}$ with data in Lebesgue or Sobolev spaces. Namely,

$$
\begin{cases}
\bar{\partial}F(z) & = & 0 & z \in D; \\[2mm]
\dot{F}(\zeta) & = & f(\zeta) & \sigma\text{-a.e. } \zeta \in bD; \\[2mm]
(F')^* & \in & L^p(bD, \sigma)
\end{cases}
\tag{6}
$$

given $f \in W^{1,p}(bD, \sigma)$. Also

$$
\begin{cases}
\bar{\partial}G & = & 0 & \text{in } D; \\[2mm]
\dfrac{\partial G}{\partial n}(\zeta) & = & g(\zeta) & \text{for } \sigma\text{-a.e. } \zeta \in bD \\[2mm]
(G')^* & \in & L^p(bD, \sigma)
\end{cases}
\tag{7}
$$

given $g \in L^p(bD, \sigma)$ subject to the compatibility condition: $\displaystyle\int_{bD} g \, d\sigma = 0$, and

$$
\begin{cases}
\bar{\partial}G & = & 0 & \text{in } D; \\[2mm]
\dfrac{\partial G}{\partial n}(\zeta) + b(\zeta)\dot{G}(\zeta) & = & r(\zeta) & \text{for } \sigma\text{-a.e. } \zeta \in bD \\[2mm]
(G')^* & \in & L^p(bD, \sigma)
\end{cases}
\tag{8}
$$

given b and r in $L^p(bD, \sigma)$, and with the Robin coefficient b subject to compatibility conditions that will be specified below. In both problems, G' denotes the complex derivative of

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G:

$$G'(z) = \frac{\partial G}{\partial z}(z), \quad z \in D$$

whereas

$$\frac{\partial G}{\partial n}(\zeta) := -iT(\zeta)(\dot{G}')(\zeta) = \lim_{\substack{z \to \zeta \\ z \in \Gamma(\zeta)}} \langle \nabla G(z), n(\zeta) \rangle_{\mathbb{R}} \quad \sigma - \text{a.e. } \zeta \in bD. \tag{9}$$

Here $\Gamma(\zeta)$ is the coordinate (or regular) cone at $\zeta \in bD$, see Definition 2.2, $T(\zeta) := t_1(\zeta) + it_2(\zeta)$ is the positively oriented, complex tangent vector at $\zeta \in bD$, and $n(\zeta) := -iT(\zeta)$ is the complex outer unit normal vector which, depending on context, we may interpret as the vector $(t_2(\zeta), -t_1(\zeta)) \in \mathbb{R}^2$.

Our candidate solution space for all of these boundary conditions is the **holomorphic Sobolev-Hardy space**

$$\mathcal{H}^{1,p}(D) := \{G \in \vartheta(D) : G' \in \mathcal{H}^p(D)\}, \quad p > 0. \tag{10}$$

It turns out that $\mathcal{H}^{1,p}(D)$ is a Banach space for any $p \geq 1$ in which case we have that

$$\mathcal{H}^{1,p}(D) \subseteq \mathcal{H}^p(D) \tag{11}$$

moreover for $1 < p < \infty$ the set inclusion (11) is also an embedding of Banach spaces, see Corollary 3.3 and Lemma 3.4. Hence for any $p \geq 1$ the members of $\mathcal{H}^{1,p}(D)$ possess non-tangential boundary values which are in $L^p(bD, \sigma)$. We write

$$h^{1,p}(bD) := \left\{ \dot{G} : \ G \in \mathcal{H}^{1,p}(D) \right\}, \quad 1 \leq p \leq \infty \tag{12}$$

and we show that $h^{1,p}(bD) \subset W^{1,p}(bD, \sigma)$, the Sobolev space for bD, see Proposition 3.6. Thus any $h \in h^{1,p}(bD)$ possesses a tangential derivative $\partial_T h$ σ-a.e. bD.

Our first main result states that $h^{1,p}(bD)$ is the optimal data space for the Regularity problem for $\overline{\partial}$. That is,

Theorem 1.2. *Let D be a bounded simply connected Lipschitz domain and let $p \geq 1$. Given $f \in W^{1,p}(bD, \sigma)$, we have that (6) is solvable if and only if $f \in h^{1,p}(bD)$, in which case (6) has a unique solution F and it lies in $\mathcal{H}^{1,p}(D)$. If $1 < p < \infty$ and $f \in h^{1,p}(bD)$ we also have that*

$$\|F^*\|_{L^\infty(bD,\sigma)} + \|(F')^*\|_{L^p(bD,\sigma)} \lesssim \|f\|_{W^{1,p}(bD,\sigma)}$$

and the implied constant depends only on D and p.

We next consider the following two subspaces of $L^p(bD, \sigma)$:

$$\mathfrak{n}^p(bD) := \left\{ g := -i\partial_T h : \ h \in h^{1,p}(bD) \right\}, \quad 1 \le p \le \infty \tag{13}$$

and

$$\mathfrak{r}_b^p(bD) := \left\{ r := -i\partial_T h + bh : h \in h^{1,p}(bD) \right\} \quad 1 \le p \le \infty. \tag{14}$$

That is

$$\mathfrak{n}^p(bD) := \left\{ g := -i\partial_T \dot{G} : \ G \in \mathcal{H}^{1,p}(D) \right\}, \quad 1 \le p \le \infty$$

and

$$\mathfrak{r}_b^p(bD) := \left\{ r := -i\partial_T \dot{H} + b\dot{H} : H \in \mathcal{H}^{1,p}(bD) \right\} \quad 1 \le p \le \infty.$$

Our second main result gives that $\mathfrak{n}^p(bD)$ is the maximal data space for the Neumann problem for the $\overline{\partial}$ operator. Specifically,

Theorem 1.3. *Let $D \subset \mathbb{C}$ be a bounded simply connected Lipschitz domain and let $p \ge 1$. Given $g \in L^p(bD, \sigma)$ we have that (7) is solvable if and only if $g \in \mathfrak{n}^p(bD)$, in which case all solutions are in $\mathcal{H}^{1,p}(D)$ and differ by an additive constant. Furthermore,*

- *the complex derivative of each solution G admits the representation*

$$G'(z) = \mathbf{C}_D(i\overline{T}g)(z), \quad z \in D.$$

Here $\mathbf{C}_D(i\overline{T}g)$ is the Cauchy integral for D acting on the pointwise product

$$i\,\overline{T}(\zeta)g(\zeta), \quad \sigma\text{-a.e. } \zeta \in bD$$

where $T(\zeta)$ is the unit tangent vector at ζ;

- *If $1 < p < \infty$ and $g \in \mathfrak{n}^p(bD)$ we have*

$$\|(G')^*\|_{L^p(bD,\sigma)} \lesssim \|g\|_{L^p(bD,\sigma)}$$

for each solution G. The implied constant depends only on D and p;

- *If $1 < p < \infty$ and $g \in \mathfrak{n}^p(bD)$, then for any $\alpha \in D$ there is a unique solution G_α in*

$$\mathcal{H}_\alpha^{1,p}(D) := \{G \in \mathcal{H}^{1,p}(D) : \ G(\alpha) = 0\}$$

and $G_\alpha \in C^{1-\frac{1}{p}}(\overline{D})$ with

$$\|G_\alpha^*\|_{C^{1-\frac{1}{p}}(\overline{D})} \lesssim \|g\|_{L^p(bD,\sigma)}.$$

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Moreover, G_α admits the representation

$$G_\alpha(z) = \mathbf{C}_D\big(h_0 - h_0(\alpha)\big)(z), \quad z \in D \tag{15}$$

where

$$h_0(\zeta) := \int\limits_{\gamma(\zeta_0,\zeta)} ig(\eta)d\sigma(\eta) \quad \in \quad h^{1,p}(bD)$$

with $\zeta_0 \in bD$ fixed arbitrarily and

$$\gamma(\zeta_0,\zeta) := \begin{cases} \text{the piece of } bD \text{ going from } \zeta_0 \text{ to } \zeta \text{ in the positive direction} & \text{if } \zeta_0 \neq \zeta, \\ \emptyset & \text{if } \zeta_0 = \zeta. \end{cases} \tag{16}$$

Furthermore, for each $\alpha \in D$ we have

$$\|G_\alpha^*\|_{L^\infty(bD,\sigma)} + \|(G_\alpha')^*\|_{L^p(bD,\sigma)} \lesssim \|g\|_{L^p(bD,\sigma)}, \quad 1 < p < \infty.$$

The implied constant depends only on D, p and α.

For the Robin problem for the $\bar\partial$ operator (8) we may take the Robin coefficient b to be in $L^p(bD,\sigma)$: this is on account of the inclusion $h^{1,p}(bD) \subset C^{1-\frac{1}{p}}(bD)$ for any $p \geq 1$ (Proposition 3.6); in particular $h^{1,p}(bD) \subset L^\infty(bD)$ so that $b\dot{G} \in L^p(bD,\sigma)$ for any $G \in \mathcal{H}^{1,p}(D)$ and $b \in L^p(bD,\sigma)$. In order to guarantee uniqueness of the solution, as well as a representation formula, we further require that

$$\int\limits_{bD} b(\xi)\, d\sigma(\xi) \neq 2k\pi \quad \text{for any} \quad k \in \mathbb{Z}. \tag{17}$$

See Example 6.1 for a holomorphic Robin problem which has infinitely many solutions where the Robin coefficient b is not identically 0 and does not satisfy condition (17). Note that the Neumann condition cannot be regarded as a sub-case of the Robin condition (namely with $b := 0$) because it does not satisfy (17).

For any ζ, $\xi \in bD$ we let $\gamma(\zeta,\xi)$ be as in (16) (with ζ and ξ in place of ζ_0 and ζ, respectively). Define the operator

$$\mathcal{T}_b r(\zeta) := \frac{i \int\limits_{bD} r(\xi)\, e^{\int_{\gamma(\zeta,\xi)} i\, b(\eta)d\sigma(\eta)}\, d\sigma(\xi)}{e^{\int_{bD} i\, b(\xi)d\sigma(\xi)} - 1}, \quad \zeta \in bD. \tag{18}$$

From (17), it is clear the denominator in (18) cannot vanish, and this makes it easy to see that

$$\mathcal{T}_b : \ L^p(bD, \sigma) \to L^\infty(bD, \sigma), \quad 1 \leq p \leq \infty$$

is bounded with

$$\|\mathcal{T}_b r\|_{L^\infty(bD,\sigma)} \leq \sigma(bD)^{1-\frac{1}{p}} \, C(D, b) \, \|r\|_{L^p(bD,\sigma)} \tag{19}$$

where $\sigma(bD)$ is the length of bD, and

$$C(D, b) := e^{\|b\|_{L^1(bD,\sigma)}} \left| e^{\int_{bD} i\, b(\xi) d\sigma(\xi)} - 1 \right|^{-1}.$$

We also have that $\mathcal{T}_b$ is smoothing for $L^p(bD, \sigma)$ whenever $1 < p < \infty$, see Proposition 6.2 for the precise statement. We may now state our third main result.

Theorem 1.4. *Let $D \subset \mathbb{C}$ be a bounded simply connected Lipschitz domain and let $p \geq 1$. Suppose that the Robin coefficient b is in $L^p(bD, \sigma)$ and satisfies condition (17). Given $r \in L^p(bD, \sigma)$ we have that (8) is solvable if and only if $r \in \mathfrak{r}_b^p(bD)$, in which case (8) has a unique solution and it lies in $\mathcal{H}^{1,p}(D)$. Furthermore,*

- *the solution G admits the representation*

$$G(z) = (\mathbf{C}_D \circ \mathcal{T}_b)\, r(z), \quad z \in D\,,$$

   where $\mathcal{T}_b$ is defined in (18);

- *for any $1 < p < \infty$ we have that*

$$\|G^*\|_{L^\infty(bD,\sigma)} + \|(G')^*\|_{L^p(bD,\sigma)} \lesssim \|r\|_{L^p(bD,\sigma)}, \tag{20}$$

   where the implied constant depends only on D, p and the Robin coefficient b.

Remark 1.5. Any bounded Lipschitz domain must be finitely connected (because locally it is a graph domain). A large portion of the harmonic analysis literature for Lipschitz domains requires the ambient domain to be simply connected: sometimes this choice is only a matter of convenience (the results can be extended to any bounded Lipschitz domain by routine arguments). But there are also results whose proofs rely on conformal mapping (this is the case for e.g., the proof of identity (1), see [16]): extending such results to arbitrary Lipschitz domains would require a detailed analysis of the boundary behavior of the Ahlfors map (the multiply-connected-domain analog of the Riemann map). To avoid these issues here we focus on simply connected domains (with a few exceptions in Section 3). Further results for multiply connected D can be found in [15, Section 5].

Remark 1.6. We expect that our results may hold in less regular settings than Lipschitz. Here we restrict the attention to the Lipschitz category because already it displays several features typical of most non-smooth frameworks e.g., the restricted p-range for the boundary value problems for the Laplacian, see Theorems 2.4 and 2.5 below, yet the proofs are less technical than those for more irregular settings. For instance, for the non-tangential approach region $\Gamma(\zeta)$, $\zeta \in bD$, here we may take the so-called *regular cone* (or *coordinate cone*) given in Definition 2.2, which is easier to work with than the corkscrew-like region $\{z \in D : |z - \zeta| \leq (1 + \beta)\operatorname{dist}(z, bD)\}$ needed to work with domains whose boundary is not a local-graph.

We henceforth use the following notation: two quantities A and B are said to satisfy $A \lesssim B$, if $A \leq CB$ for some constant $C > 0$ which may depend only on the domain D, the exponent p and, when relevant, the fixed point α, or the Robin coefficient b. We say $A \approx B$ if and only if $A \lesssim B$ and $B \lesssim A$ at the same time.

This paper is organized as follows. In Section 2 we collect a few well known features of Lipschitz domains that are relevant here, and we recall the Dirichlet and Neumann problems for the Laplace operator; we also prove Theorem 1.1. In Section 3 we present the main properties of the holomorphic Sobolev-Hardy space $\mathcal{H}^{1,p}(D)$. In Section 4 we prove Theorem 1.2 (Regularity problem for $\overline{\partial}$). Finally, Theorem 1.3 (Neumann problem for $\overline{\partial}$) and Theorem 1.4 (Robin problem for $\overline{\partial}$) are proved in Sections 5 and 6, respectively.

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2   Preliminaries

Throughout this paper the domains under consideration will be Lipschitz domains on $\mathbb{C}$, as defined below.

Definition 2.1. A bounded domain $D \subset \mathbb{C}$ with boundary bD is called a *Lipschitz domain* if there are finitely many rectangles $\{R_j\}_{j=1}^m$ with sides parallel to the coordinate axes, angles $\{\theta_j\}_{j=1}^m$, and Lipschitz functions $\phi_j : \mathbb{R} \to \mathbb{R}$ such that the collection $\{e^{-i\theta_j} R_j\}_{j=1}^m$ covers bD and $(e^{i\theta_j} D) \cap R_j = \{x + iy : y > \phi_j(x), \ x \in (a_j, b_j)\}$ for some $a_j < b_j < \infty$. We refer to such R_j's as *coordinate rectangles*.

Definition 2.2. Let D be a Lipschitz domain and fix $\beta > 0$. For any $\zeta \in bD$, let $\{\Gamma(\zeta), \zeta \in D\}$ be a family of truncated (one-sided) open cones $\Gamma(\zeta)$ with vertex at ζ satisfying the

following property: for each rectangle R_j in Definition 2.1, there exists two cones α and β, each with vertex at the origin and axis along the y axis such that for $\zeta \in bD \cap e^{-i\theta_j} R_j$,

$$e^{-i\theta_j}\alpha + \zeta \quad \subset \quad \Gamma(\zeta) \quad \subset \quad \overline{\Gamma(\zeta)} \setminus \{\zeta\} \quad \subset \quad e^{-i\theta_j}\beta + \zeta \quad \subset \quad D \cap e^{-i\theta_j} R_j.$$

It is well known that for Lipschitz D, $\Gamma(\zeta) \neq \emptyset$ for any $\zeta \in bD$; see e.g., [10] or [27, Section 0.4]. Sometimes $\Gamma(\zeta)$ is referred to as a *regular cone*, or a *coordinate cone*, see Remark 1.6.

We will need an approximation scheme of D by smooth subdomains constructed by Nečas in [24], which we refer to as a *Nečas exhaustion of D*. See also [20] and [27, Theorem 1.12]. (Recall that Lipschitz functions are differentiable almost everywhere; thus if D is a bounded Lipschitz domain, its boundary bD is a rectifiable Jordan curve that admits a (positively oriented) unit tangent vector $T(\zeta)$ for σ-a.e. $\zeta \in bD$.)

Lemma 2.3. *[24, p. 5][27, Theorem 1.12] Let D be a bounded Lipschitz domain. There exists a family $\{D_k\}_{k=1}^{\infty}$ of smooth domains with D_k compactly contained in D that satisfy the following:*

(a). For each k there exists a Lipschitz diffeomorphism Λ_k that takes D to D_k and extends to the boundaries: $\Lambda_k : bD \to bD_k$ with the property that

$$\sup\{|\Lambda_k(\zeta) - \zeta| : \zeta \in bD\} \leq C/k$$

for some fixed constant C. Moreover $\Lambda_k(\zeta) \in \Gamma(\zeta)$.

(b). There is a covering of bD by finitely many coordinate rectangles which also form a family of coordinate rectangles for bD_k for each k. Furthermore for every such rectangle R, if ϕ and ϕ_k denote the Lipschitz functions whose graphs describe the boundaries of D and D_k, respectively, in R, then $\|(\phi_k)'\|_{\infty} \leq \|\phi'\|_{\infty}$ for any k; $\phi_k \to \phi$ uniformly as $k \to \infty$, and $(\phi_k)' \to \phi'$ a.e. and in every $L^p((a,b))$ with $(a,b) \subset \mathbb{R}$ as in Definition 2.1.

(c). There exist constants $0 < m < M < \infty$ and positive functions (Jacobians) $w_k : bD \to [m, M]$ for any $k \in \mathbb{N}$, such that for any measurable set $F \subseteq bD$ and for any measurable function f_k on $\Lambda_k(F)$ the following change-of-variables formula holds:

$$\int_F f_k(\Lambda_k(\eta)) \, w_k(\zeta) \, d\sigma(\eta) = \int_{\Lambda_k(F)} f_k(\eta_k) \, d\sigma_k(\eta_k).$$

where $d\sigma_k$ denotes arc-length measure on bD_k. Furthermore we have

$$w_k \to 1 \quad \sigma\text{-a.e. } bD \quad \text{and in every} \quad L^p(bD, \sigma) \quad \text{for any} \quad 1 \leq p < \infty.$$

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(d). Let T_k denote the unit tangent vector for bD_k and T denote the unit tangent vector of bD. We have that

$$T_k \to T \quad \sigma\text{-a.e. } bD \quad \text{and in every} \quad L^p(bD, \sigma) \quad \text{for any} \quad 1 \le p < \infty.$$

(Note that in conclusions *(b)* through *(d)* the exponent $p = \infty$ cannot be allowed unless D is of class C^1.)

2.1   Boundary value problems for the Laplace operator

Let D be a simply connected Lipschitz domain. Given $F : D \to \mathbb{C}$, for any $\zeta \in bD$ we define the *non-tangential maximal function of F* as

$$F^*(\zeta) := \sup_{z \in \Gamma(\zeta)} |F(z)| \tag{21}$$

where $\Gamma(\zeta)$ is the coordinate cone given in Definition 2.2. Furthermore, define the *non-tangential limit of F* as

$$\dot{F}(\zeta) := \lim_{\substack{z \to \zeta \\ z \in \Gamma(\zeta)}} F(z). \tag{22}$$

It is known that if $F \in \mathcal{H}^p(D)$, then $\dot{F}(\zeta)$ exist for σ-a.e. $\zeta \in bD$ and $\dot{F} \in L^p(bD, \sigma)$, see [12, Theorem 10.3].

For convenience we state without proof some relevant features of two well-known boundary value problems for the Laplace equation on Lipschitz domains.

Theorem 2.4. *[19, Def. 1.7.4; Coroll. 2.1.6 & Thm 2.2.22]; [27, Corollary 3.2] Let $D \subset \mathbb{R}^k, k \ge 2$, be a bounded simply connected Lipschitz domain. Consider the* **Dirichlet problem for Laplace's operator**

$$\begin{cases} \Delta U & = \ 0 & in \ \ D; \\[2mm] \dot{U}(\zeta) & = \ u(\zeta) & for \ \sigma\text{-}a.e. \ \zeta \in bD; \\[2mm] U^* & \in \ L^p(bD, \sigma) \end{cases} \tag{23}$$

where the datum u is in $L^p(bD, \sigma)$. Then there is $\delta = \delta(D) > 0$ such that (23) is uniquely solvable whenever $u \in L^p(bD, \sigma)$ with $2 - \delta < p < \infty$. Morevoer

$$\|U^*\|_{L^p(bD, \sigma)} \lesssim \|u\|_{L^p(bD, \sigma)}.$$

Recall that the generalized normal derivative of V at $\zeta \in bD$ is

$$\frac{\partial V}{\partial n}(\zeta) := \lim_{\substack{z \to \zeta \\ z \in \Gamma(\zeta)}} \langle \nabla V(z), n(\zeta) \rangle_{\mathbb{R}} = \langle (\dot{\nabla} V)(\zeta), n(\zeta) \rangle_{\mathbb{R}} \quad \text{for} \quad \sigma - \text{a.e. } \zeta \in bD, \qquad (24)$$

see (9).

Theorem 2.5. *[21, (1.2) with $b := 0$];[19, Corollary 2.1.11 p. 48, Theorem 2.2.22 p. 56];[11, p.347] Let $D \subset \mathbb{R}^k, k \geq 2$ be a bounded simply connected Lipschitz domain. Consider the* **Neumann problem for Laplace's operator**

$$\begin{cases} \Delta V & = \quad 0 & in \;\; D; \\\\ \dfrac{\partial V}{\partial n}(\zeta) & = \quad v(\zeta) & \sigma\text{-}a.e. \; \zeta \in bD; \\\\ (\nabla V)^* & \in \quad L^p(bD, \sigma) \end{cases} \qquad (25)$$

where the datum v is in $L^p(bD, \sigma)$ and satisfies the compatibility condition

$$\int\limits_{bD} v(\zeta) \, d\sigma(\zeta) \;=\; 0.$$

Then there is $\nu = \nu(D) > 0$ such that (25) is uniquely solvable (modulo additive constants) whenever $v \in L^p(bD, \sigma)$ with $1 < p < 2 + \nu$. Moreover

$$\|(\nabla V)^*\|_{L^p(bD, \sigma)} \lesssim \|v\|_{L^p(bD, \sigma)}.$$

where the implied constant depends solely on D.

In each theorem the solution(s) admit an integral representation in terms of layer potential operators, which we omit for brevity.

Remark 2.6. The p-range in each theorem above is maximal in the Lipschitz category. See, for instance, [19, Remark 2.1.17].

2.2   The Dirichlet problem for $\overline{\partial}$: proof of Theorem 1.1

For the sake of completeness we recall the proof of Cauchy's theorem for $h^1(bD)$. Namely,

Lemma 2.7. *Let D be a bounded Lipschitz domain. Then*

$$\int_{bD} f(\zeta)\, d\zeta = 0 \qquad \text{for any} \quad f \in h^1(bD). \tag{26}$$

Proof. Let $\{D_k\}$ be a Nečas exhaustion of D and take $f \in h^1(bD)$. By definition of $h^1(bD)$, there exists $F \in \mathcal{H}^1(D)$ such that $f = \dot{F}$. For each k, D_k is compactly contained in D with smooth boundary and so by the classical Cauchy Theorem we have

$$\int_{\Lambda_k(bD)} F(\zeta_k)\, d\zeta_k = 0,$$

where Λ_k is the diffeomorphism described in Lemma 2.3 part (a). Replacing $d\zeta_k$ by $T_k(\zeta_k)\, d\sigma_k(\zeta_k)$ in the above integral and making use of the change-of-variables formula Lemma 2.3 part (c), we have

$$\int_{bD} F(\Lambda_k(\zeta)) T_k(\zeta) w_k(\zeta)\, d\sigma(\zeta) = \int_{\Lambda_k(bD)} F(\zeta_k) T_k(\zeta_k)\, d\sigma_k(\zeta_k) = \int_{\Lambda_k(bD)} F(\zeta_k)\, d\zeta_k = 0.$$

Passing $k \to \infty$ in the above and applying the dominated convergence theorem with the dominating function $MF^* \in L^1(bD, \sigma)$ (here M is a uniform bound on the w_k's see Lemma 2.3), we obtain (26). ∎

Lemma 2.8. *Let D be a bounded simply connected Lipschitz domain and $1 < p < \infty$. If $F \in \mathcal{H}^p(D)$, then*

$$\|F^*\|_{L^p(bD,\sigma)} \approx \|\dot{F}\|_{L^p(bD,\sigma)}.$$

Proof. The inequality: $\|F^*\|_{L^p(bD,\sigma)} \geq \|\dot{F}\|_{L^p(bD,\sigma)}$ is immediate from the definitions, thus we only need to show that $\|F^*\|_{L^p(bD,\sigma)} \lesssim \|\dot{F}\|_{L^p(bD,\sigma)}$ for any $1 < p < \infty$.

If $2 - \delta(D) < p < \infty$ the desired inequality follows at once from the Dirichlet problem for harmonic functions (Theorem 2.4) because $F \in \mathcal{H}^p(D)$ solves (23) with datum $u := \dot{F}$. We next suppose that $1 < p < 2$ and we let $G \in \vartheta(D)$ be any holomorphic antiderivative of F (which exists since F is holomorphic in D which is simply connected); that is

$$F(z) = G'(z), \quad z \in D.$$

It follows that $G' \in \mathcal{H}^p(D)$ with

$$|(\nabla G)^*(\zeta)| \approx (G')^*(\zeta) = F^*(\zeta), \quad \text{and} \quad (\dot{G'})(\zeta) = F(\zeta) \quad \sigma - \text{a.e. } \zeta \in bD. \tag{27}$$

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Hence

$$\frac{\partial G}{\partial n}(\zeta) = -iT(\zeta)\dot{F}(\zeta) \quad \text{for } \sigma - \text{a.e. } \zeta \in bD,$$

see (24). Thus G solves the Neumann problem for harmonic functions (25) with datum

$$v(\zeta) := -iT(\zeta)\dot{F}(\zeta) \in L^p(bD, \sigma)$$

(note that such v satisfies the compatibility condition:

$$\int_{bD} v(\zeta)d\sigma(\zeta) = 0$$

on account of Lemma 2.7.) Theorem 2.5 now ensures that $\|(\nabla G)^*\|_p \lesssim \|\dot{F}\|_p$, which proves the desired inequality by way of (27). ∎

Proof of Theorem 1.1: Let $p > 0$ and $f \in L^p(bD, \sigma)$. We first show that (4) is solvable if, and only if, $f \in h^p(bD)$. To this end, suppose that F is a solution of (4) with datum f: then it is immediate from (4), (3) and (2) that $F \in \mathcal{H}^p(D)$ and $f \in h^p(bD)$. Conversely, if $f \in h^p(bD)$, then $f = \dot{F}$ for some $F \in \mathcal{H}^p(D)$ and it follows from (2) that F solves (4). Now uniqueness for any $p > 0$ follows from [12, Theorem 10.3]. Moreover, if $p \geq 1$ and $F \in \mathcal{H}^p(D)$ is the solution of (4) with datum $f \in h^p(bD)$, then $F = \mathbf{C}_D f$ by Cauchy Formula for $\mathcal{H}^1(D)$ [12, Theorem 10.4]). Finally, if $1 < p < \infty$ then $\|F^*\|_p \lesssim \|f\|_p$ by Lemma 2.8. ∎

3 Holomorphic Sobolev-Hardy spaces

Here we highlight the main features of $\mathcal{H}^{1,p}(D)$. It is worthwhile pointing out that several of the results in this section do not require D to be simply connected: indeed, the definitions of holomorphic Hardy space $\mathcal{H}^p(D)$ and of Sobolev-Hardy space $\mathcal{H}^{1,p}(D)$, see (2) and (10), are meaningful for any bounded Lipshitz domain D (whether or not simply connected). Furthermore, since Lipschitz domains are local epigraphs, any bounded Lipschitz domain must be finitely connected. Hence, an elementary localization argument shows that any $F \in \mathcal{H}^p(D)$ has a nontangential limit $\dot{F}$ defined σ-a.e. on bD which also lies in $L^p(bD, \sigma)$ for any bounded Lipschitz domain D.

Proposition 3.1. *Let $D \subset \mathbb{C}$ be a bounded Lipschitz domain and let $\alpha \in D$. Then for any $G \in \mathcal{H}^{1,p}(D)$ and $1 \leq p \leq \infty$, we have*

$$G^*(\zeta) \lesssim (G')^*(\zeta) + \|(\dot{G}')\|_{L^p(bD,\sigma)} + |G(\alpha)| \quad \text{for any } \zeta \in bD. \tag{28}$$

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We first prove the following

Lemma 3.2. *Let D be a bounded domain with smooth boundary. Then for any $z, w \in \overline{D}$ there exists a piecewise smooth path $\gamma_w^z \subset \overline{D}$ that joins z and w, whose length does not exceed $\operatorname{diam}(D) + \sigma(bD)$.*

Proof. For any two points $z, w \in \overline{D}$, let ℓ denote the oriented line segment starting at z and ending at w. Construct a path γ_w^z that is equal to the pieces of ℓ which are in $\overline{D}$ and, for the pieces of ℓ which are outside of $\overline{D}$, γ_w^z is the portion of bD connecting the point where ℓ leaves $\overline{D}$ to the point where ℓ re-enters $\overline{D}$. Then γ_w^z has all the required properties. ■

Proof of Proposition 3.1. We claim that for any $\zeta \in bD$,

$$G(z) \lesssim (G')^*(\zeta) + \|(\dot{G}')\|_{L^p(bD,\sigma)} + |G(\alpha)| \quad \text{for all} \quad z \in \Gamma(\zeta) \tag{29}$$

from which (28) follows on account of the boundedness of D. To prove (29), let $\alpha \in D$ be fixed, and let D_α be a member of a Nečas exhaustion of D such that $\alpha \in D_\alpha$. In particular

$$D_\alpha \text{ is smooth, and } D_\alpha \cap \Gamma(\zeta) \neq \emptyset \text{ for all } \zeta \in bD.$$

Let $\zeta \in bD$ and $z \in \Gamma(\zeta)$. Then we have

$$|G(z)| \leq \int\limits_{\ell_w^z} |G'(\mu)| \, d\sigma(\mu) + |G(w)| \quad \text{for any } w \in D_\alpha \cap \Gamma(\zeta),$$

where ℓ_w^z is the line segment joining w to z. The above line integral makes sense since $\ell_w^z \subset \Gamma(\zeta) \subset D$ by convexity of the cone $\Gamma(\zeta)$ (see Definition 2.2) and $G' \in \vartheta(D)$. Moreover, the length L_1 of ℓ_w^z does not exceed the diameter of D. Hence

$$|G(z)| \leq L_1 (G')^*(\zeta) + |G(w)| \quad \text{for any} \quad z \in \Gamma(\zeta) \quad \text{and} \quad w \in D_\alpha \cap \Gamma(\zeta). \tag{30}$$

Next since D_α is smooth, for each $w \in D_\alpha$ by Lemma 3.2 there is a path $\gamma_\alpha^w \subset D_\alpha$ joining the base point α to w, whose length L_2 does not exceed some constant dependent only on D. The Cauchy integral formula for G' and Hölder inequality give

$$|G(w) - G(\alpha)| \leq \int\limits_{\gamma_\alpha^w} |G'(\mu)| \, d\sigma(\mu) \leq \frac{L_2}{2\pi} \sup_{\mu \in \gamma_\alpha^w} \int\limits_{bD} \frac{|(\dot{G}')(\eta)|}{|\eta - \mu|} d\sigma(\eta) \leq \frac{L_2 |bD|^{1-\frac{1}{p}}}{2\pi \operatorname{dist}(bD, D_\alpha)} \|(\dot{G}')\|_{L^p(bD,\sigma)},$$

for $1 \leq p \leq \infty$, whence

$$|G(w)| \lesssim |G(\alpha)| + \|(\dot{G}')\|_{L^p(bD,\sigma)} \quad \text{for any} \quad w \in D_\alpha \quad \text{and} \quad 1 \leq p \leq \infty. \tag{31}$$

The conclusion follows from (30) and (31). ■

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Note that inequality (29) is meaningful also for $G \in \mathcal{H}^{1,p}(D)$ with $0 < p < 1$, but for p in such range we are unable to prove it.

Corollary 3.3. *Let $D \subset \mathbb{C}$ be a bounded Lipschitz domain and let $\alpha \in D$. For any $G \in \mathcal{H}^{1,p}(D)$ and $1 \le p \le \infty$ we have the set inclusion*

$$\mathcal{H}^{1,p}(D) \subseteq \mathcal{H}^p(D), \quad 1 \le p \le \infty. \tag{32}$$

If in addition D is simply connected and $1 < p < \infty$, then

$$\|G^*\|_{L^p(bD,\sigma)} \lesssim |G(\alpha)| + \|G'\|_{\mathcal{H}^p(D)}. \tag{33}$$

Proof. The definition of $G \in \mathcal{H}^{1,p}(D)$ gives that $G \in \vartheta(D)$ and $G' \in \mathcal{H}^p(D)$. Consequently $(G')^*, (\dot{G}') \in L^p(bD, \sigma)$. By the pointwise estimate (28) and the boundedness of D, we get $G^* \in L^p(bD, \sigma)$ and thus $G \in \mathcal{H}^p(D)$ for any $p \ge 1$, proving (32).

If moreover, $1 < p < \infty$, then by Lemma 2.8 (which requires D to be simply connected) we also have

$$\|(G')^*\|_{L^p(bD,\sigma)} \lesssim \|(\dot{G}')\|_{L^p(bD,\sigma)} = \|G'\|_{\mathcal{H}^p(D)}.$$

Together with (28) this gives the desired inequality (33). ■

It follows from [12, Theorem 10.3] and (32) that every element of $\mathcal{H}^{1,p}(D)$ has a non-tangential limit in $h^{1,p}(bD)$ when D is a bounded simply connected Lipschitz domain, see (12). The aforementioned localization argument and (32) give that this is true also for multiply-connected D.

In view of Corollary 3.3, for any $\alpha \in D$, any $G \in \mathcal{H}^{1,p}(D)$ and any $1 \le p \le \infty$ we set

$$\|G\|_{\mathcal{H}^{1,p}(D)} := |G(\alpha)| + \|G'\|_{\mathcal{H}^p(D)}.$$

It is easy to see that the above defines a family of norms for $\mathcal{H}^{1,p}(D)$ (one norm for each choice of the base point $\alpha \in D$). The proof of Proposition 3.1 also shows that all such norms are equivalent to one other. Indeed, given $\alpha, \tilde{\alpha} \in D$ choose $k \in \mathbb{N}$ so that $\alpha, \tilde{\alpha} \in D_k$ (where D_k is a member of a Nečas exhaustion for D), then the proof leading up to (31) shows that $|G(\alpha) - G(\tilde{\alpha})| \lesssim \|(\dot{G}')\|_{L^p(bD,\sigma)} = \|G'\|_{\mathcal{H}^p(D)}$. As such we shall not specify α in the notation of $\| \cdot \|_{\mathcal{H}^{1,p}(D)}$.

Lemma 3.4. *Given D a bounded simply connected Lipschitz domain and $1 \le p \le \infty$, $\mathcal{H}^{1,p}(D)$ is a Banach space with the norm $\| \cdot \|_{\mathcal{H}^{1,p}(D)}$.*

Proof. Let G_n be a Cauchy sequence in $\mathcal{H}^{1,p}(D)$. Then G_n' is a Cauchy sequence in $\mathcal{H}^p(D)$ and thus it converges to a function F in $\mathcal{H}^p(D)$. Since D is simply connected, F has a holomorphic antiderivative G. Furthermore, we can choose an antiderivative G such that $G(\alpha) = \lim_{n \to \infty} G_n(\alpha)$. Then $G \in \mathcal{H}^{1,p}(D)$ and G_n converges to G in the $\mathcal{H}^{1,p}(D)$-norm. ∎

Remark 3.5. The space $\mathcal{H}^{1,p}(D)$ is not closed in the $\mathcal{H}^p(D)$-norm, already for $D = \mathbb{D}$ (the unit disc). To see this, let $F \in \mathcal{H}^p(\mathbb{D})$ be such that $F' \notin \mathcal{H}^p(\mathbb{D})$ (any $F \in \mathcal{H}^p(\mathbb{D})$ that does not continuously extend to $\overline{\mathbb{D}}$ will satisfy this condition; see, for example, [12, Theorem 3.11]). Let F_n be the nth Taylor polynomial of F centered at 0: clearly $F_n \in \mathcal{H}^{1,p}(\mathbb{D})$ (F_n is a holomorphic polynomial). Moreover, since $F \in \mathcal{H}^p(\mathbb{D})$, F_n converges to F in the $\mathcal{H}^p(\mathbb{D})$-norm. So $F \in \overline{\mathcal{H}^{1,p}(\mathbb{D})}^{\mathcal{H}^p} \setminus \mathcal{H}^{1,p}(\mathbb{D})$, proving that $\mathcal{H}^{1,p}(\mathbb{D})$ is not closed in the $\mathcal{H}^p$-norm.

Let $W^{1,p}(bD, \sigma)$, $p \geq 1$, be the **Sobolev space** consisting of all $L^p(bD, \sigma)$ functions whose first order derivatives along the tangential direction of bD exist in the distributional sense and are in $L^p(bD, \sigma)$. Here we shall adopt the definition of the tangential derivative in [27] that is invariant under rotations and translations. To be precise, given a coordinate rectangle $R = (a, b) \times (c, d)$ and Lipschitz boundary function ϕ for D as in Definition 2.1 with angle $\theta = 0$ (for simplicity and without loss of generality),

- [27, Definition 1.7] we say that a *real-valued* function $u \in W^{1,p}(bD, \sigma)$ if $u \in L^p(bD, \sigma)$, and there exists $f \in L^p(R \cap bD, d\sigma)$ such that

$$-\int_a^b u\,(t, \phi(t))\,\psi'(t)dt = \int_a^b f\,(t, \phi(t))\,\psi(t)dt \text{ for all } \psi \in C_c^\infty((a, b)); \qquad (34)$$

- [27, Definition 1.9] the *tangential derivative* $\partial_T u(\zeta)$ of u at σ-a.e. $\zeta := (t, \phi(t)) \in bD$ is defined by

$$\partial_T u(\zeta)T(\zeta) := (f(\zeta), 0) - \langle (f(\zeta), 0), n(\zeta) \rangle_{\mathbb{R}} n(\zeta) = \langle (f(\zeta), 0), T(\zeta) \rangle_{\mathbb{R}} T(\zeta)$$

where $n(\zeta)$ is the outer unit normal vector at $\zeta \in bD$. Thus,

$$\partial_T u(\zeta) := \langle (f(\zeta), 0), T(\zeta) \rangle_{\mathbb{R}}.$$

In our local coordinates the unit tangent vector at $\zeta := (t, \phi(t))$ is

$$T(\zeta) = \frac{(1, \phi'(t))}{|\,(1, \phi'(t))\,|},$$

hence

$$\partial_T u(\zeta) = \frac{f(\zeta)}{|\,(1, \phi'(t))\,|}, \quad \zeta := (t, \phi(t)), \quad t \in (a, b). \qquad (35)$$

- A *complex-valued* function $u + iv$ is said to be in $W^{1,p}(bD, \sigma)$ if both its real part u and its imaginary part v are in $W^{1,p}(bD, \sigma)$, in which case we let

$$\partial_T(u + iv)(\zeta) = \partial_T u(\zeta) + i\partial_T v(\zeta).$$

- $W^{1,p}(bD, \sigma)$ is a Banach Space with

$$\|f\|_{W^{1,p}(bD,\sigma)} := \|f\|_{L^p(bD,\sigma)} + \|\partial_T f\|_{L^p(bD,\sigma)}$$

Proposition 3.6. *Let D be a bounded simply connected Lipschitz domain. The following properties hold:*

(a). $h^{1,p}(bD) \subset W^{1,p}(bD, \sigma)$ for any $p \geq 1$. That is, for any $G \in \mathcal{H}^{1,p}(D)$, the tangential derivative $\partial_T \dot{G}$ is a regular distribution and satisfies

$$\partial_T \dot{G}(\zeta) = T(\zeta)(\dot{G'})(\zeta) \quad \sigma\text{-a.e. } \zeta \in bD. \tag{36}$$

(b). $h^{1,p}(bD) \subset C^{1-\frac{1}{p}}(bD)$ for any $1 \leq p < \infty$. More precisely, for any $G \in \mathcal{H}^{1,p}(D)$ there is a unique $g \in C^{1-\frac{1}{p}}(bD)$ such that $\dot{G}(\zeta) = g(\zeta)$ for $\sigma\text{-a.e. } \zeta \in bD$.

Recall that for a closed set X we say that $g \in C^{1-\frac{1}{p}}(X)$ if its Hölder norm, defined by

$$\|g\|_{C^{1-\frac{1}{p}}(X)} := \sup_{x \in X} |g(x)| + \sup_{x,y \in X} \frac{|g(x) - g(y)|}{|x - y|^{1-\frac{1}{p}}},$$

is finite. When $p = 1$, it reduces to the continuous function space $C(X)$.

Proof of Proposition 3.6: For part (a), let $\{R_j\}_{j=1}^m$ be a family of coordinate rectangles covering bD as in Lemma 2.3 part (b); for any member of such family we henceforth omit the label j and write $R = (a, b) \times (c, d)$ as well as θ, ϕ, ϕ_k as in Definition 2.1 and Lemma 2.3. For any testing function $\psi \in C_c^\infty((a, b))$, we have

$$-\int_a^b \dot{G}(t, \phi(t)) \, \psi'(t) dt = -\lim_{k \to \infty} \int_a^b G(t, \phi_k(t)) \, \psi'(t) dt$$

$$= \lim_{k \to \infty} \int_a^b \left(G_x(t, \phi_k(t)) + G_y(t, \phi_k(t)) (\phi_k)'(t) \right) \psi(t) dt$$

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Here the first equality is due to Lebesgue's dominated convergence theorem (where we use that $G^* \in L^p(bD, \sigma) \subseteq L^1(bD, \sigma)$); the second equality follows from integration by parts. Since G is holomorphic in D, by the Cauchy-Riemann equation for holomorphic functions, we have for $t \in (a, b)$,

$$G_x(t, \phi_k(t)) = G'(t, \phi_k(t)) \quad \text{and} \quad G_y(t, \phi_k(t)) = iG'(t, \phi_k(t)).$$

Thus

$$-\int_a^b \dot{G}(t, \phi(t))\, \psi'(t) dt = \lim_{k \to \infty} \int_a^b G'(t, \phi_k(t)) \left(1 + i (\phi_k)'(t)\right) \psi(t) dt$$

$$= \int_a^b (\dot{G'})(t, \phi(t)) (1 + i\phi'(t)) \psi(t) dt,$$

where the last equality is due to Lemma 2.3 parts (b) and again Lebesgue's dominated convergence theorem (where this time we use that $(G')^* \in L^p(bD, \sigma) \subseteq L^1(bD, \sigma)$).

Hence (34) holds with

$$f(t, \phi(t)) := (\dot{G'})(t, \phi(t)) (1 + i\phi'(t)).$$

By (35) and the fact that in complex notation we have that $T = \frac{1+i\phi'}{|1+i\phi'|}$ on $R \cap bD$, it follows that for σ-a.e. $\zeta = (t + i\phi(t)) \in bD$,

$$\partial_T \dot{G}(\zeta) = \frac{(\dot{G'})(\zeta)(1 + i\phi'(t))}{|1 + i\phi'(t)|} = T(\zeta)(\dot{G'})(\zeta).$$

Conclusion (a) is thus proved.

For the proof of conclusion (b), we let $\{D_k\}$ be a Nečas exhaustion $\{D_k\}$ of D, see Lemma 2.3. Given $G \in \mathcal{H}^{1,p}(D)$ we fix a point $\nu_0 \in bD$ where $\dot{G}(\nu_0)$ exists (since $G \in \mathcal{H}^{1,p}(D) \subseteq \mathcal{H}^p(D)$, $\dot{G}$ exists σ-a.e.). Let $\zeta \in bD$ be arbitrary and let $\gamma(\nu_0, \zeta)$ be the arc in bD from ν_0 to ζ oriented in the positive direction, see (16). Since G is holomorphic on D and $\Lambda_k(\gamma(\nu_0, \zeta))$ is a smooth curve contained in D, by the Fundamental Theorem of Calculus for line integrals

we have

$$
\begin{aligned}
G(\Lambda_k(\zeta)) &= G(\Lambda_k(\nu_0)) + \int\limits_{\Lambda_k(\gamma(\nu_0,\zeta))} \nabla_{T_k} G(\eta_k)\, d\sigma_k(\eta_k) \\
&= G(\Lambda_k(\nu_0)) + \int\limits_{\Lambda_k(\gamma(\nu_0,\zeta))} T_k(\eta_k)\, G'(\eta_k)\, d\sigma_k(\eta_k) \\
&= G(\Lambda_k(\nu_0)) + \int\limits_{\gamma(\nu_0,\zeta)} (T_k \circ \Lambda_k)(\eta)\, (G' \circ \Lambda_k)(\eta)\, w_k(\eta)\, d\sigma(\eta),
\end{aligned}
\tag{37}
$$

where the last identity is due to Lemma 2.3. Now for any $\zeta \in bD$ the right-hand side converges as $k \to \infty$ by the Lebesgue's dominated convergence theorem (again as $(G')^* \in L^p(bD,\sigma) \subseteq L^1(bD,\sigma)$), and by Lemma 2.3, to

$$
g(\zeta) := \dot{G}(\nu_0) + \int\limits_{\gamma(\nu_0,\zeta)} T(\eta)\, (\dot{G'})(\eta)\, d\sigma(\eta).
$$

It follows from the above that for any $\zeta \in bD$ the left hand side of (37) also converges as $k \to \infty$, and in fact

$$
g(\zeta) = \lim_{k \to \infty} G(\Lambda_k(\zeta)) = \dot{G}(\zeta) \quad \text{for } \sigma\text{-a.e. } \zeta \in bD,
$$

because $\Lambda_k(\zeta) \in \Gamma(\zeta)$, see Lemma 2.3. Now Hölder's inequality gives

$$
|g(\zeta) - g(\xi)| \le \|(\dot{G'})\|_{L^p(bD,\sigma)} \left(\sigma(\gamma_\xi^\zeta) \right)^{1-\frac{1}{p}}.
$$

But

$$
\sigma(\gamma_\xi^\zeta) \lesssim |\zeta - \xi| \quad \text{for any } \zeta, \xi \in bD
$$

since Lipschitz domains are chord-arc, and the implied constant depends only on D (see, for example, Section 7.4 in [25]). Hence $g \in C^{1-\frac{1}{p}}(bD)$; the proof is concluded. ∎

Remark 3.7. In the case when $1 < p < \infty$, Proposition 3.6 *(b)* can be directly obtained by incorporating Proposition 3.6 *(a)* with the Sobolev embedding theorem ([23, Theorem 3.6.6]). In fact, as a result of this, for any $g \in h^{1,p}(bD)$, one has $g \in W^{1,p}(bD,\sigma) \subset C^{1-\frac{1}{p}}(bD)$ along with the following estimate

$$
\|g\|_{C^{1-\frac{1}{p}}(bD)} \lesssim \|g\|_{W^{1,p}(bD,\sigma)}.
$$

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4 The Regularity problem for $\overline{\partial}$: proof of Theorem 1.2

Let $f \in W^{1,p}(bD, \sigma)$. We first show that (6) is solvable if, and only if, $f \in h^{1,p}(bD)$. To this end, suppose that F is a solution of (6) with datum f: then F is holomorphic, and it is immediate from (6), (2) and (10) that $F' \in \mathcal{H}^p(D)$ which means that $F \in \mathcal{H}^{1,p}(D)$, and therefore $f \in h^{1,p}(bD)$, see (12). Conversely, if $f \in h^{1,p}(bD)$, then $f = \dot{F}$ for some $F \in \mathcal{H}^{1,p}(D)$ and (10) gives that F solves (6). Uniqueness follows from Theorem 1.1 (since $\mathcal{H}^{1,p}(D) \subset \mathcal{H}^p(D)$).

If $p > 1$ and $F \in \mathcal{H}^{1,p}(D)$ is the solution of (6) with datum $f \in h^{1,p}(bD)$, then $F = \mathbf{C}_D f$ by Theorem 1.1. Make use of Remark 3.7 and the boundedness of the operator $\mathbf{C}_D$ from $C^{1-\frac{1}{p}}(bD)$ to $C^{1-\frac{1}{p}}(\overline{D})$ (see [22, Theorem 3.3]). Namely, the holomorphic function $\mathbf{C}_D f$ extends to $F \in C^{1-\frac{1}{p}}(\overline{D})$ with

$$\|F\|_{C^{1-\frac{1}{p}}(\overline{D})} \lesssim \|f\|_{C^{1-\frac{1}{p}}(bD)} \lesssim \|f\|_{W^{1,p}(bD,\sigma)}.$$

In particular,
$$\|F^*\|_{L^\infty(bD,\sigma)} \lesssim \|f\|_{W^{1,p}(bD,\sigma)}.$$

On the other hand, we also have that F' solves (4) with datum

$$\overline{T}(\zeta)\, \partial_T f(\zeta)$$

which belongs to $h^p(bD)$ by Proposition 3.6(a), hence

$$\|(F')^*\|_{L^p(bD,\sigma)} \lesssim \|\overline{T}\, \partial_T f\|_{L^p(bD,\sigma)} = \|\partial_T f\|_{L^p(bD,\sigma)}$$

again by Theorem 1.1. ∎

5 The Neumann problem for $\overline{\partial}$: proof of Theorem 1.3

Let $p \geq 1$ and $g \in L^p(bD, \sigma)$. We first show that the holomorphic Neumann problem (7) is solvable if and only if $g \in \mathfrak{n}^p(bD)$. To see this, suppose that G is a solution to (7) with datum g: then it is immediate from (7) and (10) that $G \in \mathcal{H}^{1,p}(D)$, hence $\dot{G} \in h^{1,p}(bD)$. The latter together with (7) again, (9), (36) and (13) give that $g \in \mathfrak{n}^p(bD)$. Conversely, if $g \in \mathfrak{n}^p(bD)$ then $g = -i\partial_T \dot{G}$ for some $G \in \mathcal{H}^{1,p}(D)$ and it follows from (9), (36), (10) and (2) that G solves (7) with datum g. Now [12, Theorem 10.3] gives that G' is unique, hence G is unique modulo additive constants. Next we note that if $G \in \mathcal{H}^{1,p}(D)$ is a solution of (7)

with datum $g \in \mathfrak{n}^p(bD)$, then $F := G' \in \mathcal{H}^p(D)$ is the solution of the holomorphic Dirichlet problem (4) with datum

$$f(\zeta) := i\overline{T}(\zeta)g(\zeta) = i\overline{T}(\zeta)\frac{\partial G}{\partial n}(\zeta) = (\dot{G}')(\zeta)$$

by (7) and (9). Hence $f \in h^p(bD)$ and Theorem 1.1 gives $G' = \mathbf{C}_D(i\overline{T}g)$.

Theorem 1.1 also grants that

$$\|(G')^*\|_{L^p(bD,\sigma)} \lesssim \|i\overline{T}g\|_{L^p(bD,\sigma)} = \|g\|_{L^p(bD,\sigma)} \qquad 1 < p < \infty.$$

Furthermore, it is clear from all we did that for any $\alpha \in D$, the holomorphic Neumann problem (7) has a unique solution $G_\alpha \in \mathcal{H}_\alpha^{1,p}(D)$.

For the remainder of the proof, we will assume that $1 < p < \infty$. Fix one point $\zeta_0 \in bD$ and define

$$h_0(\zeta) = \int\limits_{\gamma(\zeta_0,\zeta)} ig(\eta)d\sigma(\eta) \quad \text{for every} \quad \zeta \in bD.$$

where $\gamma(\zeta_0,\zeta)$ is as in (16). Let $\xi \in bD$ be a Lebesgue point of g. Since $g \in L^p(bD,\sigma) \subseteq L^1(bD,\sigma)$, Lebesgue points exist σ-a.e. on bD. We have for any $\zeta \in bD$,

$$\left| \frac{h_0(\zeta) - h_0(\xi)}{\int_{\gamma(\xi,\zeta)} d\sigma} - ig(\xi) \right| = \left| \frac{\int_{\gamma(\xi,\zeta)} (g(\eta) - g(\xi)) d\sigma(\eta)}{\int_{\gamma(\xi,\zeta)} d\sigma} \right|.$$

The right-hand side of the above equation tends to 0 as ζ approaches ξ along bD since ξ is a Lebesgue point of g. Thus, h_0 admits a tangential derivative σ-a.e. and moreover

$$\partial_T h_0 = ig. \tag{38}$$

Hölder's inequality gives $\|h_0\|_\infty \lesssim \|g\|_p$; it follows that $h_0 \in W^{1,p}(bD,\sigma)$ with

$$\|h_0\|_{W^{1,p}(bD,\sigma)} \lesssim \|g\|_{L^p(bD,\sigma)}.$$

Furthermore, since $g \in \mathfrak{n}^p$, we also have $h_0 \in h^{1,p}(bD)$ by (38) and the definition of $\mathfrak{n}^p$, along with the fact that every function in $W^{1,p}(bD,\sigma)$ with vanishing tangential derivative is constant. See, for instance, [23, Theorem 3.6.5]. Now [22, Theorem 3.3] together with the Sobolev embedding $W^{1,p}(bD,\sigma) \subset C^{1-\frac{1}{p}}(bD)$ in Remark 3.7, gives that $\mathbf{C}_D(h_0) \in C^{1-\frac{1}{p}}(\overline{D})$ with

$$\|\mathbf{C}_D(h_0)\|_{C^{1-\frac{1}{p}}(\overline{D})} \lesssim \|h_0\|_{C^{1-\frac{1}{p}}(bD)} \lesssim \|h_0\|_{W^{1,p}(bD,\sigma)} \lesssim \|g\|_{L^p(bD,\sigma)}. \tag{39}$$

Let G_α be the unique solution to (7) with datum g, hence $G_\alpha(\alpha) = 0$ and $\partial_T \dot{G}_\alpha = ig$ by (9) and (36), and (38) along with [23, Theorem 3.6.5] give that $\dot{G}_\alpha = h_0 + \lambda_\alpha$ for some constant λ_α. That is, G_α solves the Regularity problem (6) with datum

$$h_0 + \lambda_\alpha \in h^{1,p}(bD).$$

(Note that $\lambda_\alpha \in h^{1,p}(bD)$ since $H(z) := \lambda_\alpha z \in \mathcal{H}^{1,p}(D)$, see (12) and (2).)

Theorem 1.2 now gives

$$G_\alpha(z) = \mathbf{C}_D h_0(z) + \lambda_\alpha, \quad z \in D \tag{40}$$

(recall that $\mathbf{C}_D(1)(z) \equiv 1$). Since $G_\alpha(\alpha) = 0$, we further have $\lambda_\alpha = -\mathbf{C}_D h_0(\alpha)$, hence G_α admits the representation (15). In particular, (39) gives that $\lambda_\alpha \lesssim \|g\|_{L^p(bD,\sigma)}$ and that $G_\alpha \in C^{1-\frac{1}{p}}(\overline{D})$ with

$$\|G_\alpha\|_{C^{1-\frac{1}{p}}(\overline{D})} \lesssim \|g\|_{L^p(bD,\sigma)}.$$

In particular,

$$\|G_\alpha^*\|_{L^\infty(bD,\sigma)} \lesssim \|g\|_{L^p(bD,\sigma)}.$$

■

Remark 5.1. The above construction does not depend on the choice of ζ_0 in the following sense. If another point ζ_1 were chosen instead with a corresponding function h_1, then h_0 and h_1 would only differ by an additive constant that can be chosen so that $h_0(\zeta) - h_0(\alpha) = h_1(\zeta) - h_1(\alpha)$ for σ-a.e. $\zeta \in bD$.

6 The Robin problem for $\overline{\partial}$

We begin with an example that shows that the uniqueness of the holomorphic Robin problem fails in general if the compatibility condition (17) is dropped.

Example 6.1. Consider the following homogeneous holomorphic Robin problem on $\mathbb{D}$:

$$\begin{cases} \overline{\partial} G & = 0 \quad \text{in } \mathbb{D}; \\[2mm] \dfrac{\partial G}{\partial n}(\zeta) - \dot{G}(\zeta) & = 0 \quad \text{for } \zeta \in b\mathbb{D} \\[2mm] (\nabla G)^* \in L^p(b\mathbb{D}) \end{cases} \tag{41}$$

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Making use of the fact that $T(\zeta) = i\zeta$ on $b\mathbb{D}$ one can directly verify that $G(z) := Cz$ solves (41) for any constant C. Note that $b = -1$ on $b\mathbb{D}$ with

$$\int_{b\mathbb{D}} b(\zeta)d\sigma(\zeta) = -2\pi.$$

6.1 The smoothing operator $\mathcal{T}_b$

For $\zeta, \xi \in bD$ with $\zeta \neq \xi$, recall that $\gamma(\zeta, \xi)$ is the piece of bD joining ζ to ξ in the positive direction. In particular, we define $\gamma(\zeta, \zeta) = \emptyset$; if ξ approaches ζ along the positive orientation of γ, we denote it by $\xi \to \zeta^-$, with $\gamma(\zeta, \zeta^-) = bD$.

Proposition 6.2. *Let D be a bounded simply connected Lipschitz domain and let $1 < p < \infty$. Assume $b \in L^p(bD, \sigma)$ satisfies the compatibility condition (17). Then for the operator $\mathcal{T}_b$ defined in (18), we have that*

(i.) $\mathcal{T}_b$ *is bounded:* $L^p(bD, \sigma) \to C^{1-\frac{1}{p}}(bD)$. *Namely*

$$\|\mathcal{T}_b r\|_{C^{1-\frac{1}{p}}(bD)} \lesssim \|r\|_{L^p(bD,\sigma)}.$$

(ii.) $\mathcal{T}_b$ *is bounded:* $L^p(bD, \sigma) \to W^{1,p}(bD, \sigma)$. *Namely*

$$\|\mathcal{T}_b r\|_{L^p(bD,\sigma)} + \|\partial_T(\mathcal{T}_b r)\|_{L^p(bD,\sigma)} \lesssim \|r\|_{L^p(bD,\sigma)}, \tag{42}$$

and $\mathcal{T}_b r$ *gives the unique solution to*

$$-i\partial_T h(\zeta) + b(\zeta)h(\zeta) = r(\zeta) \quad for \quad \sigma - a.e. \ \zeta \in bD. \tag{43}$$

(iii.) $\mathcal{T}_b$ *takes* $\mathfrak{r}^p(bD)$ *into* $h^{1,p}(bD)$. *Hence* $\mathbf{C}_D \circ \mathcal{T}_b$ *takes* $\mathfrak{r}^p(bD)$ *into* $\mathcal{H}^{1,p}(D)$.

Proof. We adopt the shorthand

$$\tilde{b}_z(\zeta) := e^{i \int_{\gamma(z,\zeta)} b(\xi)d\sigma(\xi)} \quad \text{for} \quad \zeta \in bD;$$

$$\tilde{b}_0 := e^{i \int_{bD} b(\xi)d\sigma(\xi)},$$

and

$$\tilde{r}(z) := \int_{bD} \tilde{b}_z(\zeta)r(\zeta)d\sigma(\zeta), \quad z \in bD.$$

Then the compatibility condition (17) is equivalent to

$$\tilde{b}_0 - 1 \neq 0. \tag{44}$$

Proof of (i.): We only need to show

$$\|\tilde{r}\|_{C^{1-\frac{1}{p}}(bD)} \lesssim \|r\|_{L^p(bD,\sigma)}. \tag{45}$$

First, since $\|\tilde{b}_z\|_{L^\infty(bD,\sigma)} \leq e^{\|b\|_{L^1(bD,\sigma)}} \lesssim 1$,

$$\sup_{\zeta \in bD} |\tilde{r}(\zeta)| \lesssim \|r\|_{L^1(bD,\sigma)} \lesssim \|r\|_{L^p(bD,\sigma)}. \tag{46}$$

For fixed $\zeta_1, \zeta_2 \in bD$, the arclength $\sigma(\gamma(\zeta_1,\zeta_2)) \approx |\zeta_1 - \zeta_2|$ by the Lipschitz property of bD. Without loss of generality, assume that $|\zeta_1 - \zeta_2|$ is small. For σ-a.e. $\xi \in bD \setminus \gamma(\zeta_1,\zeta_2)$,

$$\left| e^{i \int_{\gamma(\zeta_1,\xi)} b(\eta)d\sigma(\eta)} - e^{i \int_{\gamma(\zeta_2,\xi)} b(\eta)d\sigma(\eta)} \right| \leq e^{\|b\|_{L^1(bD,\sigma)}} \left| \int_{\gamma(\zeta_1,\zeta_2)} |b(\eta)|d\sigma(\eta) \right| \lesssim |\zeta_1 - \zeta_2|^{1-\frac{1}{p}}.$$

Here we used the mean-value theorem in the first inequality (more precisely, $|e^{z_1} - e^{z_2}| \leq e^x |z_1 - z_2|$ whenever $|z_1|, |z_2| < x := \|b\|_{L^1(bD,\sigma)}$) and Hölder inequality in the second inequality. On the other hand, for σ-a.e. $\xi \in \gamma(\zeta_1,\zeta_2)$,

$$\left| e^{i \int_{\gamma(\zeta_k,\xi)} b(\eta)d\sigma(\eta)} \right| \leq e^{\|b\|_{L^1(bD,\sigma)}} \lesssim 1, k = 1, 2.$$

Thus by Hölder inequality

$$|\tilde{r}(\zeta_1) - \tilde{r}(\zeta_2)| \lesssim \int_{bD\setminus\gamma(\zeta_1,\zeta_2)} \left| e^{i \int_{\gamma(\zeta_1,\xi)} b(\eta)d\sigma(\eta)} - e^{i \int_{\gamma(\zeta_2,\xi)} b(\eta)d\sigma(\eta)} \right| |r(\xi)|d\sigma(\xi)$$

$$+ \int_{\gamma(\zeta_1,\zeta_2)} |r(\xi)|d\sigma(\xi)$$

$$\lesssim |\zeta_1 - \zeta_2|^{1-\frac{1}{p}} \|r\|_{L^p(bD,\sigma)}.$$

Equation (45) follows from the above inequality and (46).

Proof of (ii.): We first show that there exists a unique solution to (43). We shall adopt the arclength variable $s \in [0, s_0)$ to parametrize bD, where $\zeta(0) = \lim_{s \to s_0^-} \zeta(s) = z$ for a fixed $z \in bD$, and $|\zeta'(s)| = 1$ for a.e. $s \in [0, s_0)$.

Write $h(s) := h(\zeta(s))$, and similarly for T, b, $\tilde{b}_z$ and r. In particular, for $s \in [0, s_0)$, we have $\tilde{b}_z(s) = e^{i \int_0^s b(s)ds}$. It is immediate to verify that $\tilde{b}_z, \tilde{b}_z^{-1} \in W^{1,p}((0, s_0))$. We further continuously extend $\tilde{b}_z$ to s_0 with $\tilde{b}_z(s_0) := e^{i \int_0^{s_0} b(s)ds} = \tilde{b}_0$.

Since $\partial_T h = \frac{dh}{ds}$, any solution h to (43) necessarily satisfies

$$-ih'(s) + b(s)h(s) = r(s), \text{ for a.e. } s \in (0, s_0). \tag{47}$$

This is a first order linear ordinary differential equation. Using the method of integrating factors, we have

$$\frac{d}{ds}\left(\tilde{b}_z(s)h(s)\right) = i\tilde{b}_z(s)r(s), \text{ for a.e. } s \in (0, s_0). \tag{48}$$

Thus any solution to (47) is of the form

$$h(s) = i\tilde{b}_z^{-1}(s) \int_0^s \tilde{b}_z(t)r(t)dt + C\tilde{b}_z^{-1}(s) \text{ for a.e. } s \in (0, s_0) \tag{49}$$

for some constant C. Since $\tilde{b}_z r \in L^p((0, s_0))$, one has $\int_0^s \tilde{b}_z(t)r(t)dt \in W^{1,p}((0, s_0)) \subset L^\infty((0, s_0))$. Together with the fact that $\tilde{b}_z^{-1} \in W^{1,p}((0, s_0))$, one can further verify that $h \in W^{1,p}((0, s_0))$. In particular, by the Sobolev embedding theorem, every solution to (47), and hence to (43), is continuous on $(0, s_0)$. Note that due to the closedness of bD, a continuous solution to (47) becomes a solution to (43) if and only if the following natural boundary value condition is imposed to (47):

$$h(0) = h(s_0). \tag{50}$$

Making use of the facts that $\tilde{b}_z(0) = 1$ and $\tilde{b}_z(s_0) = \tilde{b}_0$, (50) is further equivalent to

$$i + C = i\tilde{b}_0 \int_0^{s_0} \tilde{b}_z(t)r(t)dt + C\tilde{b}_0. \tag{51}$$

Due to (44), the constant C in (49) is uniquely determined by (51). As a consequence, there exists a unique solution to (47) with boundary value condition (50), and thus a unique solution to (43).

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Next, we verify that $h = \mathcal{T}_b r$ in (18) solves (43) explicitly. Integrate (48) from 0 to s_0 to get

$$\tilde{b}_z(s_0)h(s_0) - \tilde{b}_z(0)h(0) = \int_0^{s_0} i\tilde{b}_z(s)r(s)ds = i\int_{bD} \tilde{b}_z(\zeta)r(\zeta)d\sigma(\zeta).$$

Making use of the facts that $h(s_0) = h(0) = h(z)$, $\tilde{b}_z(0) = 1$ and $\tilde{b}_z(s_0) = \tilde{b}_0$ again, we see that $\mathcal{T}_b r$ solves (43).

Finally we show the desired estimate for $\mathcal{T}_b$. The estimate $\sup_{\zeta \in bD} |\mathcal{T}_b r(\zeta)|$ is done in (19). For the estimate of $\partial_T(\mathcal{T}_b r)$, we make use of the fact that $\mathcal{T}_b r$ satisfies (43) and (19) to get

$$
\begin{aligned}
\|\partial_T(\mathcal{T}_b r)\|_{L^p(bD,\sigma)} &\leq \|b\mathcal{T}_b r\|_{L^p(bD,\sigma)} + \|r\|_{L^p(bD,\sigma)} \\
&\lesssim \|b\|_{L^p(bD,\sigma)} \sup_{\zeta \in bD} |\mathcal{T}_b r(\zeta)| + \|r\|_{L^p(bD,\sigma)} \\
&\lesssim \|b\|_{L^p(bD,\sigma)} \|r\|_{L^p(bD,\sigma)} + \|r\|_{L^p(bD,\sigma)} \lesssim \|r\|_{L^p(bD,\sigma)}.
\end{aligned}
\tag{52}
$$

Conclusion (ii.) is thus proved.

Proof of (iii.): We only need to prove that $\mathcal{T}_b \mathfrak{r}^p(bD) \subset h^{1,p}(bD)$. Given $r \in \mathfrak{r}^p(bD)$, by definition of $\mathfrak{r}^p(bD)$ there exists some $h \in h^{1,p}(bD)$ such that h satisfies (43). Due to the uniqueness of solutions to (43) as proved in part (ii.), $\mathcal{T}_b r$ must be equal to h and thus $\mathcal{T}_b r \in h^{1,p}(bD)$. ∎

6.2 Proof of Theorem 1.4

Let $p \geq 1$ and $r \in L^p(bD, \sigma)$. We first show that the holomorphic Robin problem (8) is solvable if and only if $r \in \mathfrak{r}^p(bD)$. To see this, suppose that G is a solution to (8) with datum r: then it is immediate from (8) and (10) that $G \in \mathcal{H}^{1,p}(D)$, hence $\dot{G} \in h^{1,p}(bD)$. Moreover (8), (9) and (36) give that

$$-i\partial_T \dot{G}(\zeta) + b(\zeta)\dot{G}(\zeta) = r(\zeta) \quad \text{for} \quad \sigma - \text{a.e.} \ \zeta \in bD, \tag{53}$$

proving that $r \in \mathfrak{r}^p(bD)$, see (14). Conversely, if $r \in \mathfrak{r}^p(bD)$ then it satisfies (53) for some $G \in \mathcal{H}^{1,p}(D)$, see (14) and (12), and it follows from (53), (9), (36), (10) and (2) that G solves (8) with datum r.

Next, we observe that if $G \in \mathcal{H}^{1,p}(D)$ is a solution of (8) with datum $r \in \mathfrak{r}^p(bD)$ then G is also a solution of the regularity problem (6) with datum

$$f(\zeta) := \mathcal{T}_b r(\zeta)$$

(which is in $h^{1,p}(bD)$ by Proposition 6.2 (iii.)). Hence the uniqueness and the representation formula: $G = (\mathbf{C}_D \circ \mathcal{T}_b)r$ follow from Theorem 1.2.

We are left to prove the estimate (20) for $1 < p < \infty$. Since $\mathcal{T}_b r \in C^{1-\frac{1}{p}}(bD)$ by Proposition 6.2 (i.), one has $G := (\mathbf{C}_D \circ \mathcal{T}_b)r \in C^{1-\frac{1}{p}}(\overline{D}) \cap \vartheta(D)$ with

$$\|G\|_{C^{1-\frac{1}{p}}(\overline{D})} \lesssim \|\mathcal{T}_b r\|_{C^{1-\frac{1}{p}}(bD)}.$$

In particular, by the above and (45)

$$\|G^*\|_{L^\infty(bD,\sigma)} \le \sup_{z \in D} |G| \lesssim \|\mathcal{T}_b r\|_{C^{1-\frac{1}{p}}(bD)} \lesssim \|r\|_{L^p(bD,\sigma)}.$$

Moreover, since $(G')^* \in L^p(bD,\sigma)$ by assumption, we have $G' \in \mathcal{H}^p(D)$. Thus by Lemma 2.8, (36) and (52)

$$\|(G')^*\|_{L^p(bD,\sigma)} \lesssim \|(\dot{G}')\|_{L^p(bD,\sigma)} = \|\partial_T \mathcal{T}_b r\|_{L^p(bD,\sigma)} \lesssim \|r\|_{L^p(bD,\sigma)}.$$

∎

Remark 6.3. As an immediate consequence of Theorem 1.4, the data space for the holomorphic Robin problem can be characterized equivalently as

$$\mathfrak{r}_b^p(bD) = \left\{ r \in L^p(bD,\sigma) : \mathcal{T}_b r \in h^{1,p}(bD) \right\},$$

where $\mathcal{T}_b r$ is given by (18). On the other hand, from the proof of Proposition 6.2, given $r \in \mathfrak{r}_b^p(bD)$, the unique solution G also lies in $C^{1-\frac{1}{p}}(\overline{D})$.

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